# Game Theory

## Variable-sum games: Non-cooperative games

As said, variable-sum games can be classified into two types:
a. Non-cooperative
b. Co-operative

Let us look at non-cooperative games first.

### 1. The concept of equilibrium and solution methods

The concept of equilibrium is still that of Nash equilibrium:

$$E_A (x^*, y^*) \geq E_A (x, y^*)$$

$$E_B (x^*, y^*) \geq E_B (x^*, y)$$

However, unlike in zero-sum games, the MAXIMIN criterion will **not** produce an equilibrium except by chance. The main reason is that in a variable-sum payoff situation, **maximizing one's payoff is NOT the same as minimizing the other's payoff**.

Let us check with a simple example:

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_1$ | (2, 2) | (3, 3) |
| $A_2$ | (1, 1) | (4, 4) |

| | A | B |
|---|---|---|
| Maximin strategy | $A_1(2)$ | $B_2(3)$ |

But ($A_1$, $B_2$) is not an equilibrium pair (of strategies) because A has an incentive to change from strategy $A_1$ to $A_2$. Which is the equilibrium pair in this case?

Hence we need other solution methods. Moreover, in the zero-sum case, there may be more than one equilibria with the same value. In the variable-sum non-cooperative case, however, there may be multiple equilibria with different values. In the case of “prisoner’s dilemma” the concepts of equilibrium and optimality are also divorced.

* In other words, there are **suboptimal** equilibrium or equilibria among the multiple equilibria.

## 2. Solution method: SWASTIKA

Let us look at an example: the battle of the sexes

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_1$ | (1, 4) | (0, 0) |
| $A_2$ | (0, 0) | (4, 1) |

A: husband
B: wife
1: shopping
2: watching football

Or, to turn it into an economic example:

A: rural population  1: build an airport
B: urban population  2: build a railway

The SWASTIKA method of solution:

Let x, 1-x be the frequencies assigned to $A_1$, $A_2$ respectively
y, 1-y be the frequencies assigned to $B_1$, $B_2$ respectively

Then the expected payoff for A: $E_A$ and B: $E_B$ will be:

$$E_A = 1xy + 0x(1-y) + 0(1-x)y + 4(1-x)(1-y)$$
$$= 5xy + 4 - 4x - 4y$$
$$= x(5y-4) + 4 - 4y \quad \text{--------------------- (1)}$$

$$E_B = 4xy + 0x(1-y) + 0(1-x)y + 1(1-x)(1-y)$$
$$= 5xy + 1 - x - y$$
$$= y(5x-1) + 1 - x \quad \text{--------------------- (2)}$$

Now (1) is the expected payoff function for A in terms of the strategies adopted by A and B, and (2) is the expected payoff function for B in terms of the strategies adopted by A and B. obviously, A has to choose x carefully to maximize $E_A$, taking into account what B may do with y, and vice versa.

A's perspective

Now given: $E_A = \underbrace{x(5y-4)}_{\text{A can influence the outcome}} + \underbrace{4 - 4y}_{\text{Independent of A's action}}$

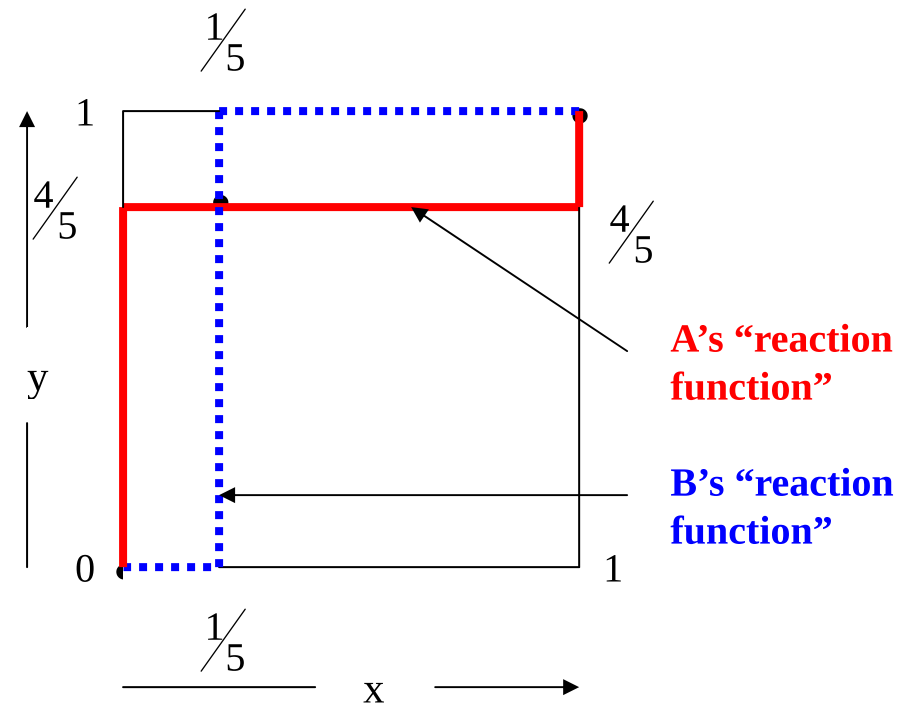
To max. $E_A$

If $y < \frac{4}{5} \rightarrow x = 0$

If $y = \frac{4}{5} \rightarrow 0 \leq x \leq 1$

If $y > \frac{4}{5} \rightarrow x = 1$

Graphically:



B's perspective

Given: $E_B = y(5x-1) + \underbrace{1 - x}_{\text{independent}}$

To max. $E_B$

If $x < \frac{1}{5} \rightarrow y = 0$

If $x = \frac{1}{5} \rightarrow 0 \leq y \leq 1$

If $x > \frac{1}{5} \rightarrow y = 1$

So there are three equilibria:

1. $x = 0$, $y = 0$, equilibrium pair: $(A_2, B_2)$, solution value $V(4, 1)$

2. $x = 1$, $y = 1$, equilibrium pair: $(A_1, B_2)$, solution value $V(1, 4)$

3. $x = \frac{1}{5}$, $y = \frac{4}{5}$, equilibrium pair:

$((\frac{1}{5}, \frac{4}{5})(\frac{4}{5}, \frac{1}{5}))$

What is the solution value V(?)

Use the $E_A$ and $E_B$ functions.

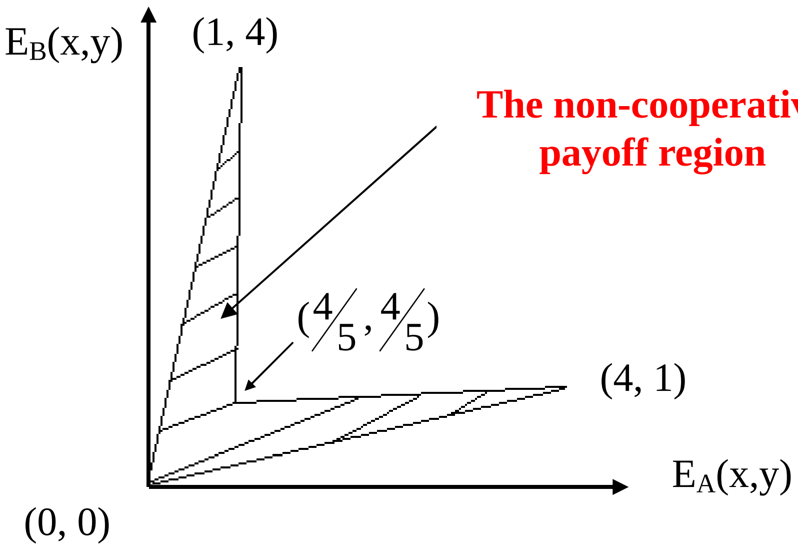
$E_A = x\,(5y - 4) + 4 - 4y$

(with $x = \frac{1}{5}$, $y = \frac{4}{5}$, $y = \frac{4}{5}$)

$E_B = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$V(\frac{4}{5}, \frac{4}{5})$

We can plot the multiple equilibria in a graph of $E_A - E_B$ space:




This non-cooperative payoff region is obtained by varying the values of x and y randomly, i.e. with co-ordination between A and B.

## 3. The prisoner's dilemma: socially suboptimal equilibrium

**The "prisoner's dilemma" is the most famous example in variable sum non-cooperative games.** In general form, it has the form of the following payoff matrix:

| | |
|---|---|
| (c, c) | (a, d) |
| (d, a) | (b, b) |

where $a > b > c > d$

or, alternatively

| | |
|---|---|
| (c, c) | (a, d) |
| (d, a) | (b, b) |

$\Rightarrow$

| | |
|---|---|
| (b, b) | (d, a) |
| (a, d) | (c, c) |

still $a > b > c > d$.

e.g. Two prisoners, A and B, who have jointly committed a crime, are interrogated separately by the police. For both, strategy 1 is to admit the crime, and strategy 2 is not to admit. The payoffs are the numbers of years in prison (so the more the worse). The police have insufficient evidence. Hence if both A and B do not admit the crime, i.e. ($A_2$, $B_2$), the court can only sentence each to 2 years of imprisonment.

Then the police tell each of them: if he admits the crime and the other does not, there will already be sufficient evidence. The police will ask for mercy for him in the court and condemn the other. He will get one year and the other will get four years in jail. And vice versa:

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_1$ | (3, 3) | (1, 4) |
| $A_2$ | (4, 1) | (2, 2) |

We can also solve the prisoner's dilemma by the SWASTIKA method. Remember that the payoffs represent number of years of imprisonment, so the objective of both is to **MINIMIZE** his payoff. Look at matrix 1

$$E_A = 3xy + 1x(1-y) + 4(1-x)y + 2(1-x)(1-y)$$
$$= 2 - x + 2y \qquad \text{---------------------} \ (1)$$

$$E_B = 3xy + 4x(1-y) + 1(1-x)y + 2(1-x)(1-y)$$
$$= 2 + 2x - y \qquad \text{---------------------} \ (2)$$

From (1) and (2), it is clear that A cannot do anything to offset y in minimizing $E_A$, nor can B do anything to offset x in minimizing $E_B$.

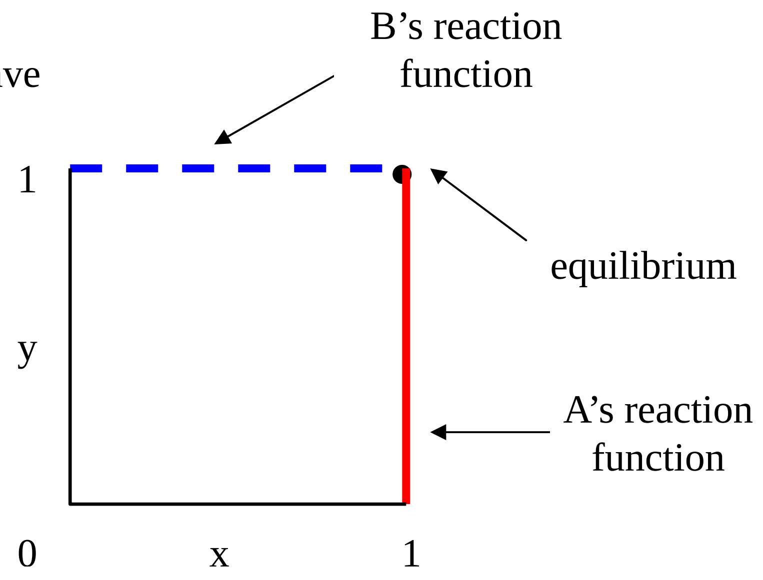
So, given - x in (1); min $E_A \rightarrow x = 1$

Given -y in (2); min $E_B \rightarrow y = 1$

Hence **the equilibrium pair is** ($A_1$, $B_1$) and V(3, 3) is the value of the game, which is **socially suboptimal**.

Check that ($A_2$, $B_2$) with V(2, 2) are socially optimal, but not equilibrium solutions.

Graphically, we have




The prisoner's dilemma shows that in a situation of social conflict, it is possible to result in a suboptimal equilibrium, from which both parties, motivated by self interest, cannot escape.

One interesting question is this: if A and B are allowed to discuss and "collude", can they escape the "dilemma"?

The answer is no! Why?

The **solution to restore social optimality** may then be:

(1) **external intervention**: morality, law, government policy or force.

(2) **"learning to cooperate"** if the game is repeated infinite number of times". See Varian, chapter 28, pp. 496-498.