Game Theory

Zero-sum games (零和遊戲)

Also called “constant sum games”: my gain, your loss; and vice versa: your gain, my loss. Hence, there is a fundamental conflict of interests between any party to the game. Such games are by nature non-cooperative.

1. For the sake of simplicity, we restrict ourselves to two-player zero-sum games. A typical example is market share game – the division of the 100% in market share.

<table>
<thead>
<tr>
<th>A's strategies</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>20</td>
<td>80</td>
<td>25</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>A2</td>
<td>40</td>
<td>35</td>
<td>50</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>A3</td>
<td>60</td>
<td>65</td>
<td>25</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

This is just A's payoff matrix. What about B's? Because the total payoff for both A and B is always 100%, so B's payoff matrix can be constructed by deducting each of the payoff by 100. However, there is no urgency to construct B's payoff. We can simply analyze the game by assuming that:

1. A aims to maximize the payoff in his own matrix
2. B aims to minimize the payoff in A’s matrix

Since A's gain is B's loss, so if B minimizes A's payoff by choosing the right strategy, he maximizes his own! This is in the nature of zero-sum (or constant sum) games.

Now we come to the difficult part of the game theory; how should A and B choose their optimal strategy respectively? Which criterion (maximin, maximax, minimax regret, maximum likelihood, and expected value) should be used? And suppose they choose different criteria, how and when will the game reach an “equilibrium”?
2. Concept of equilibrium in games

We have discussed the concept of equilibrium in the case of constrained optimization. In games, the most important and popularly accepted concept of equilibrium is "Nash equilibrium" (Nash was a Nobel Prize winner in economics in 1994).

Varian on p.471 gives the following common sense description of "Nash equilibrium":

“… a pair of strategies is a Nash equilibrium if A's choice is optimal, given B's choice, and B's choice is optimal given A's choice. … a Nash equilibrium can be interpreted as a pair of expectations about each person's choice such that, when the other person's choice is revealed, neither individual wants to change his behaviour.”

More formally, we can define the "equilibrium pair" of strategy under Nash equilibrium as \((x^*, y^*)\),

where \(x^*\) is the strategy chosen by A
\(y^*\) is the strategy chosen by B.

Then Nash equilibrium is a state where

\[
E_A (x^*, y^*) \geq E_A (x, y^*)
\]

\[
E_B (x^*, y^*) \geq E_B (x^*, y)
\]

Where \(E_A(\bullet)\) and \(E_B(\bullet)\) are the expected payoffs for A and B respectively given the chosen pair of strategies \(\bullet\).

When a Nash equilibrium is reached, it does not pay for A to change his strategy unilaterally (if B does not change his) nor does it pay for B to change his strategy unilaterally (if A does not change his). In other words, a Nash equilibrium is a situation where no player has any incentive to change his strategy unilaterally.
Take a crude example, suppose A's payoff matrix for another market share game is the following:

\[
\begin{array}{c|cc}
\text{A's matrix} & \text{B}_1 & \text{B}_2 \\
\hline
\text{A}_1 & 50 & 60 \\
\text{A}_2 & 40 & 50 \\
\end{array}
\quad
\begin{array}{c|cc}
\text{B's matrix} & \text{B}_1 & \text{B}_2 \\
\hline
\text{A}_1 & 50 & 40 \\
\text{A}_2 & 60 & 50 \\
\end{array}
\]

We can check that \((A_1, B_1)\) is a Nash equilibrium pair of strategies, but \((A_2, B_2)\) is not. Why?

Answer: Given \((A_1, B_1)\), if A changes his strategy unilaterally from \(A_1\) to \(A_2\) (but B sticks with \(B_1\)), A's payoff will drop from 50% to 40% of market share. Likewise, given \((A_1, B_1)\) if B changes his strategy from \(B_1\) to \(B_2\) (which A keeps \(A_1\)), B's payoff will fall from 50% to 40%. Therefore \((A_1, B_1)\) is an equilibrium pair of strategies under the concept of Nash equilibrium. It is "a state of persistence" under which both players in the game have no incentives to change their strategies.

3. Solving zero-sum games

Let us go back to the market share game on p.1

\[
\begin{array}{c|cccc}
\text{A's strategies} & \text{B}_1 & \text{B}_2 & \text{B}_3 & \text{B}_4 \\
\hline
\text{A}_1 & 20 & 80 & 25 & 70 \\
\text{A}_2 & 40 & 35 & 50 & 35 \\
\text{A}_3 & 60 & 65 & 25 & 10 \\
\end{array}
\]

How do we find the "Nash equilibrium" in this two-person, constant-sum game of market shares?

The first step is to locate the dominant strategy and to eliminate the dominated or inferior strategy. (See Varian, p.470). A dominant
strategy is one which a player will choose no matter what the other player does. A **dominated or inferior** strategy is one which a player will never choose whatever the other player does.

In fact, we can use this rule to find the Nash equilibrium for the game on the top of p.3.

How should we use the **dominant** strategy to find the Nash equilibrium for

<table>
<thead>
<tr>
<th></th>
<th>B₁</th>
<th>B₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>A₂</td>
<td>40</td>
<td>50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>B₁</th>
<th>B₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>A₂</td>
<td>60</td>
<td>50</td>
</tr>
</tbody>
</table>

Now, more importantly, it can be proved that for two-person, zero-sum games, where **pure strategies** can achieve equilibrium, both players must adopt the **MAXIMIN** criterion. But pure strategies MAY not produce an equilibrium.

In [A]:  
A₁ (min) : 50  
A₂(min) : 40  \( \therefore \) maximin => A₁

In [B]:  
B₁ (min) : 50  
B₂(min) : 40  \( \therefore \) maximin => B₁

Hence, if both A and B adopt the **MAXIMIN** criterion: the equilibrium pair of strategies will be (A₁, B₁).

(Mathematically, it can be proved that for all zero-sum games, a Nash equilibrium will be achieved if both sides adopt the **MAXIMIN** criterion in decision making.) You should check why A₁ dominates A₂ for A, and B₁ dominates B₂ for B. Given that A₁ and B₁ dominate, they, i.e. (A₁, B₁) must constitute an "equilibrium pair" (of strategies) – "Nash"-wise of course.

Now, let us eliminate the dominated strategies from the 3 x 4 market share matrix: i.e. "simplifying it".
To recap:

**A's payoff matrix**

<table>
<thead>
<tr>
<th>B's strategies</th>
<th>B_1</th>
<th>B_2</th>
<th>B_3</th>
<th>B_4</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A's Strategies</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A_1</td>
<td>20</td>
<td>80</td>
<td>25</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>A_2</td>
<td>40</td>
<td>35</td>
<td>50</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>A_3</td>
<td>60</td>
<td>65</td>
<td>25</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Since this is A's payoff matrix, A will start thinking which strateg(ies) that B will never use. Therefore,

i. From B's perspective, he will choose (minimizing)
   - if A_1 → B_1
   - if A_2 → B_2 or B_4
   - if A_3 → B_4

therefore B_3 is dominated and will never be used. So we have to delete the whole B_3 column. The payoff matrix for A becomes

<table>
<thead>
<tr>
<th></th>
<th>B_1</th>
<th>B_2</th>
<th>B_3</th>
<th>B_4</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1</td>
<td>20</td>
<td>80</td>
<td>25</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>A_2</td>
<td>40</td>
<td>35</td>
<td>50</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>A_3</td>
<td>60</td>
<td>65</td>
<td>25</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

ii. Then we shift to A's perspective (maximizing)
   - if B_1 → A_3
   - if B_2 → A_1
   - if B_4 → A_1

therefore A_2 is dominated and eliminated.
iii. Going to B's perspective (minimizing)

<table>
<thead>
<tr>
<th></th>
<th>B₁</th>
<th>B₂</th>
<th>B₄</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>20</td>
<td>80</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>A₃</td>
<td>60</td>
<td>65</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

if A₁ → B₁

if A₃ → B₄

Therefore, B₂ is eliminated.

iv. So we have

<table>
<thead>
<tr>
<th></th>
<th>B₁</th>
<th>B₄</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>20</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>A₃</td>
<td>60</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Now we cannot find any more dominant or dominated strategy.

* We can check that none of the four possible pairs of strategies \([(A₁, B₁), (A₁, B₄), (A₃, B₁), (A₃, B₄)]\) satisfies the Nash equilibrium concept of

\[
E_A (x^*, y^*) \geq E_A (x, y^*)
\]

\[
E_B (x^*, y^*) \geq E_B (x^*, y)
\]

** Important: a Nash equilibrium is broken if any party has an incentive to change the strategy, even if the other does not have.

O.K., now, given the non-reducible matrix

<table>
<thead>
<tr>
<th></th>
<th>B₁</th>
<th>B₄</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>20</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>A₃</td>
<td>60</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

It is clear that there is no equilibrium pair of “pure” strategies.
4. Solving zero-sum games by defining mixed strategies for "cleaned" payoff matrix

In games where pure strategies cannot achieve a Nash equilibrium, we have to look at **mixed strategies**. (See Varian, pp.472-3)

Pure strategy: each agent is making one choice and sticking to it once and forever.

Mixed strategy: each agent “randomizes” his strategies – to assign a frequency to each strategy and to play his choice according to it: e.g. A may choose to play $A_1$ 50% of the time and $A_3$ 50% of the time.

Now, for the reduced matrix. We can define the frequency/probability distribution for both A and B:

- $p$: the proportion of time that A chooses $A_1$
- $1-p$: the proportion of time that A chooses $A_3$
- $q$: the proportion of time that B chooses $B_1$
- $1-q$: the proportion of time that B chooses $B_4$.

What is an **optimal mixed strategy**? It is the one that gives the player an expected payoff which is invariant to the strategy (pure or mixed) adopted by the other player.

So given

\[
\begin{array}{c|cc}
(p) & (q) B_1 & (1-q) B_4 \\
\hline
A_1 & 20 & 70 \\
A_3 & 60 & 10 \\
\end{array}
\]

From A's perspective:

If B chooses $B_1$, A's expected payoff is
\[
E_A = 20p + 60(1-p) \quad (1)
\]

If B chooses $B_4$, A's expected payoff is
\[
E_A = 70p + 10(1-p) \quad (2)
\]
To fulfill the "invariance" requirement, set (1) = (2)

\[ 20p + 60(1-p) = 70p + 10(1-p) \]
\[ 20p + 60 - 60p = 70p + 10 - 10p \]
\[ 100p = 50 \]
\[ p = 0.5 \quad (50\%) \]
\[ 1-p = 0.5 \quad (50\%) \]

Therefore A's strategy is to use A₁ and A₃ 50% of the time, no matter what B does.

Then from B's perspective:

If A chooses A₁, B's expected payoff is
\[ 80q + 30(1-q) \]
\[ \text{----------------------------- (3)} \]

If A chooses A₃, B's expected payoff is
\[ 40q + 90(1-q) \]
\[ \text{----------------------------- (4)} \]

Set (3) = (4) for "invariance":
\[ 80q + 30(1-q) = 40q + 90(1-q) \]
\[ 80q + 30 - 30q = 40q + 90 - 90q \]
\[ 100q = 60 \]
\[ q = 0.6 \quad (60\%) \]
\[ 1-q = 0.4 \quad (40\%) \]

Therefore B's strategy is to use B₁ 60% of the time and B₄ 40% of the time.

Now, given that
A's strategy
\[ p \quad \text{--- 0.5 for A₁} \]
\[ 1-p \quad \text{--- 0.5 for A₃} \]

B's strategy
\[ q \quad \text{--- 0.6 for B₁} \]
\[ 1-q \quad \text{--- 0.4 for B₄} \]

the expected payoffs for A and B will respectively be:
\[
E_A = 0.5(20) + 0.5(60) = 40 / E_A = 0.5(70) + 0.5(10) = 40
\]

\[
E_B = 0.6(80) + 0.4(30) = 60 / E_B = 0.6(40) + 0.4(90) = 60
\]

Therefore, \[E_A = 40\]
\[E_B = 60\]

The “equilibrium value” \( V \) is \((40, 60)\).

In sum, the solutions to this zero-sum game are as follows:
1. The “equilibrium pair” (of mixed strategies) is \((0.5, 0.5) (0.6, 0.4))\).
2. The “equilibrium value” (of payoffs) is \((40, 60)\).

*The convention is always to write the strategies and payoff for A first, then B’s.