Game Theory

Zero-sum games (零和遊戲)

Also called “constant sum games”: my gain, your loss; and vice versa: your gain, my loss. Hence, there is a fundamental conflict of interests between any party to the game. Such games are by nature **non-cooperative**.

1. **For the sake of simplicity, we restrict ourselves to two-player zero-sum games.** A typical example is market share game – the division of the 100% in market share.

e.g. A's payoff matrix

		B's strategies				
		B_1	B_2	B_3	B_4	%
A's	A_1	20	80	25	70	
Strategies	A_2	40	35	50	35	
	A_3	60	65	25	10	

This is just A's payoff matrix. What about B's? Because the total payoff for both A and B is always 100%, so B's payoff matrix can be constructed by deducting each of the payoff by 100. However, there is no urgency to construct B's payoff. We can simply analyze the game by assuming that:
1. A aims to **maximize** the payoff **in his own matrix**
2. B aims to **minimize** the payoff **in A’s matrix**

Since A's gain is B's loss, so if B minimizes A's payoff by choosing the right strategy, he maximizes his own! This is in the nature of zero-sum (or constant sum) games.

Now we come to the difficult part of the game theory; how should A and B choose their optimal strategy respectively? Which criterion (maximin, maximax, minimax regret, maximum likelihood, and expected value) should be used? And suppose they choose different criteria, how and when will the game reach an “equilibrium”?

2. Concept of equilibrium in games

We have discussed the concept of equilibrium in the case of constrained optimization. In games, the most important and popularly accepted concept of equilibrium is "Nash equilibrium" (Nash was a Nobel Prize winner in economics in 1994).

Varian on p.471 gives the following common sense description of **"Nash equilibrium"**:

"… a pair of strategies is a Nash equilibrium if A's choice is optimal, given B's choice, and B's choice is optimal given A's choice. … a Nash equilibrium can be interpreted as a pair of expectations about each person's choice such that, when the other person's choice is revealed, neither individual wants to change his behaviour."

More formally, we can define the "equilibrium pair" of strategy under Nash equilibrium as (x*, y*),
where x* is the strategy chosen by A
y* is the strategy chosen by B.

Then Nash equilibrium is a state where

$$E_A(x^*, y^*) \geq E_A(x, y^*)$$

$$E_B(x^*, y^*) \geq E_B(x^*, y)$$

Where $E_A(\bullet)$ and $E_B(\bullet)$ are the expected payoffs for A and B respectively given the chosen pair of strategies $(\bullet)$.

When a Nash equilibrium is reached, it does not pay for A to change his strategy unilaterally (if B does not change his) nor does it pay for B to change his strategy unilaterally (if A does not change his). **In other words, a Nash equilibrium is a situation where no player has any incentive to change his strategy unilaterlly.**

Take a crude example, suppose A's payoff matrix for another market share game is the following:

A's matrix	B_1	B_2 %
A_1	50	60
A_2	40	50

B's matrix	B_1	B_2 %
A_1	50	40
A_2	60	50

We can check that (A_1, B_1) is a Nash equilibrium pair of strategies, but (A_2, B_2) is not. Why?

Answer: Given (A_1, B_1), if A changes his strategy unilaterally from A_1 to A_2 (but B sticks with B_1), A's payoff will drop from 50% to 40% of market share. Likewise, given (A_1, B_1) if B changes his strategy from B_1 to B_2 (which A keeps A_1), B's payoff will fall from 50% to 40%. Therefore (A_1, B_1) is an equilibrium pair of strategies under the concept of Nash equilibrium. It is "a state of persistence" under which both players in the game have no incentives to change their strategies.

3. Solving zero-sum games

Let us go back to the market share game on p.1

A's payoff matrix

		B's strategies			
		B_1	B_2	B_3	B_4 %
A's Strategies	A_1	20	80	25	70
	A_2	40	35	50	35
	A_3	60	65	25	10

How do we find the "Nash equilibrium" in this two-person, constant-sum game of market shares?

The first step is **to locate the dominant strategy and to eliminate the dominated or inferior strategy**. (See Varian, p.470). A dominant strategy is one which a player will choose no matter what the other player does. A **dominated or inferior** strategy is one which a player will never choose whatever the other player does.

In fact, we can use this rule to find the Nash equilibrium for the game on the top of p.3.

How should we use the dominant strategy to find the Nash equilibrium for

[A]	B_1	B_2
A_1	50	60
A_2	40	50

[B]	B_1	B_2
A_1	50	40
A_2	60	50

?

Now, more importantly, it can be proved that for two-person, zero-sum games, where **pure strategies** can achieve equilibrium, both players must adopt the MAXIMIN criterion. But pure strategies MAY not produce an equilibrium.

In [A]: A_1 (min) : 50

A_2(min) : 40 $\therefore$ maximin => A_1

In [B]: B_1 (min) : 50

B_2(min) : 40 $\therefore$ maximin => B_1

Hence, if both A and B adopt the **MAXIMIN** criterion: the equilibrium pair of strategies will be (A_1, B_1).

(Mathematically, it can be proved that for all zero-sum games, a Nash equilibrium will be achieved if both sides adopt the **MAXIMIN** criterion in decision making.) You should check why A_1 dominates A_2 for A, and B_1 dominates B_2 for B. Given that A_1 and B_1 dominate, they, i.e. (A_1, B_1) must constitute an "equilibrium pair" (of strategies) – "Nash"-wise of course.

Now, let us eliminate the dominated strategies from the 3 x 4 market share matrix: i.e. "simplifying it".

To recap:

A's payoff matrix

		B's strategies				
		B_1	B_2	B_3	B_4	%
A's	A_1	20	80	25	70	
Strategies	A_2	40	35	50	35	
	A_3	60	65	25	10	

Since this is A's payoff matrix, A will start thinking which strateg(ies) that B will never use. Therefore,

i. From B's perspective, he will choose (minimizing)

if $A_1 \rightarrow B_1$

if $A_2 \rightarrow B_2$ or B_4

if $A_3 \rightarrow B_4$

therefore B_3 is dominated and will never be used. So we have to delete the whole B_3 column. The payoff matrix for A becomes

	B_1	B_2	~~B_3~~	B_4	%
A_1	20	80	~~25~~	70	
A_2	40	35	~~50~~	35	
A_3	60	65	~~25~~	10	

ii. Then we shift to A's perspective (maximizing)

if $B_1 \rightarrow A_3$

if $B_2 \rightarrow A_1$

if $B_4 \rightarrow A_1$

therefore A_2 is dominated and eliminated.

iii. Going to B's perspective (minimizing)

	B_1	B_2	B_4	%
A_1	20	80	70	
A_3	60	65	10	

if $A_1 \rightarrow B_1$

if $A_3 \rightarrow B_4$

Therefore, B_2 is eliminated.

iv. So we have

	B_1	B_4	%
A_1	20	70	
A_3	60	10	

Now we cannot find any more dominant or dominated strategy.
* We can check that none of the four possible pairs of strategies [(A_1, B_1), (A_1, B_4), (A_3, B_1), (A_3, B_4)] satisfies the Nash equilibrium concept of

$$E_A(x^*, y^*) \geq E_A(x, y^*)$$

$$E_B(x^*, y^*) \geq E_B(x^*, y)$$

** Important: a Nash equilibrium is broken if any party has an incentive to change the strategy, even if the other does not have.
O.K., now, given the non-reducible matrix

	B_1	B_4	%
A_1	20	70	
A_3	60	10	

It is clear that there is no equilibrium pair of "pure" strategies.

4. Solving zero-sum games by defining mixed strategies for "cleaned" payoff matrix

In games where pure strategies cannot achieve a Nash equilibrium, we have to look at **mixed strategies**. (See Varian, pp.472-3)
Pure strategy: each agent is making one choice and sticking to it once and forever.

Mixed strategy: each agent "randomizes" his strategies – to assign a frequency to each strategy and to play his choice according to it: e.g. A may choose to play A_1 50% of the time and A_3 50% of the time.

Now, for the reduced matrix. We can define the frequency/probability distribution for both A and B:

p: the proportion of time that A chooses A_1
1-p: the proportion of time that A chooses A_3

q: the proportion of time that B chooses B_1
1-q: the proportion of time that B chooses B_4.

What is an optimal mixed strategy? It is the one that gives the player an expected payoff which is invariant to the strategy (pure or mixed) adopted by the other player.

So given

	(q) B_1	(1-q) B_4
(p) A_1	20	70
(1-p) A_3	60	10

From A's perspective:

If B chooses B_1, A's expected payoff is
$E_A = 20p + 60(1-p)$ ------------------------------------ (1)

If B chooses B_4, A's expected payoff is
$E_A = 70p + 10(1-p)$ ------------------------------------ (2)

To fulfill the "invariance" requirement, set (1) = (2)

$$20p + 60(1-p) = 70p + 10(1-p)$$
$$20p + 60 – 60p = 70p + 10 – 10p$$
$$100p = 50$$
$$p = 0.5 \quad (50\%)$$
$$1-p = 0.5 \quad (50\%)$$

Therefore A's strategy is to use A_1 and A_3 50% of the time, no matter what B does.

Then from B's perspective:

If A chooses A_1, B's expected payoff is

$80q + 30(1-q)$ ------------------------------------ (3)

If A chooses A_3, B's expected payoff is

$40q + 90(1-q)$ ------------------------------------ (4)

Set (3) = (4) for "invariance":

$$80q + 30(1-q) = 40q + 90(1-q)$$
$$80q + 30 – 30q = 40q + 90 – 90q$$
$$100q = 60$$
$$q = 0.6 \quad (60\%)$$
$$1-q = 0.4 \quad (40\%)$$

Therefore B's strategy is to use B_1 60% of the time and B_4 40% of the time.

Now, given that

A's strategy

p --- 0.5 for A_1

$1-p$ --- 0.5 for A_3

B's strategy

q --- 0.6 for B_1

$1-q$ --- 0.4 for B_4

the expected payoffs for A and B will respectively be:

$E_A = 0.5(20) + 0.5(60) = 40$ / $E_A = 0.5(70) + 0.5(10) = 40$

$E_B = 0.6(80) + 0.4(30) = 60$ / $E_B = 0.6(40) + 0.4(90) = 60$

Therefore, $E_A = 40$
$E_B = 60$

The "equilibrium value" V is (40, 60).

In sum, the solutions to this zero-sum game are as follows:
1. The **"equilibrium pair"** (of mixed strategies) is ((0.5, 0.5) (0.6, 0.4)).
2. The **"equilibrium value"** (of payoffs) is (40, 60).

*The convention is always to write the strategies and payoff for A first, then B's.