From non-cooperative variable-sum games to voting theory

Game Theory

Variable-sum games: Non-cooperative games

As said, variable-sum games can be classified into two types:

a. Non-cooperative

b. Cooperative

Let us look at non-cooperative games first.

1. The concept of equilibrium and solution methods

The concept of equilibrium is still that of Nash equilibrium:

$$E_A(x^*, y^*) \geq E_A(x, y^*)$$

$$E_B(x^*, y^*) \geq E_B(x^*, y)$$

However, unlike in zero-sum games, the MAXIMIN criterion will **not** produce an equilibrium except by chance. The main reason is that in a variable-sum payoff situation, **maximizing one's payoff is NOT the same as minimizing the other's payoff**.

Let us check with a simple example:

	B_1	B_2
A_1	(2, 2)	(3, 3)
A_2	(1, 1)	(4, 4)

	A	B
Maximin strategy	A_1(2)	B_2(3)

But (A_1, B_2) is not an equilibrium pair (of strategies) because A has an incentive to change from strategy A_1 to A_2. Which is the equilibrium pair in this case?

Hence we need other solution methods. Moreover, in the zero-sum case, there may be more than one equilibria with the same value. In the variable-sum non-cooperative case, however, there may be multiple equilibria with different values. In the case of “prisoner’s dilemma” the concepts of equilibrium and optimality are also divorced.

* In other words, there are **suboptimal** equilibrium or equilibria among the multiple equilibria.

2. Solution method: SWASTIKA

Let us look at an example: the battle of the sexes

	B_1	B_2
A_1	(1, 4)	(0, 0)
A_2	(0, 0)	(4, 1)

A: husband
B: wife
1: shopping
2: watching football

Or, to turn it into an economic example:

A: rural population
B: urban population

1: build an airport
2: build a railway

The SWASTIKA method of solution:

Let x, 1-x be the frequencies assigned to A_1, A_2 respectively
y, 1-y be the frequencies assigned to B_1, B_2 respectively

Then the expected payoff for A: E_A and B: E_B will be:

$$
\begin{aligned}
E_A &= 1xy + 0x(1-y) + 0(1-x)y + 4(1-x)(1-y) \\
&= 5xy + 4 - 4x - 4y \\
&= x(5y-4) + 4 - 4y \qquad \text{--------------------} \ (1)
\end{aligned}
$$

$$
\begin{aligned}
E_B &= 4xy + 0x(1-y) + 0(1-x)y + 1(1-x)(1-y) \\
&= 5xy + 1 - x - y \\
&= y(5x-1) + 1 - x \qquad \text{--------------------} \ (2)
\end{aligned}
$$

Now (1) is the expected payoff function for A in terms of the strategies adopted by A and B, and (2) is the expected payoff function for B in terms of the strategies adopted by A and B. obviously, A has to choose x carefully to maximize E_A, taking into account what B may do with y, and vice versa.

A's perspective

Now given: $E_A = \underbrace{x(5y-4)}_{\text{A can influence the outcome}} + \underbrace{4 - 4y}_{\text{Independent of A's action}}$

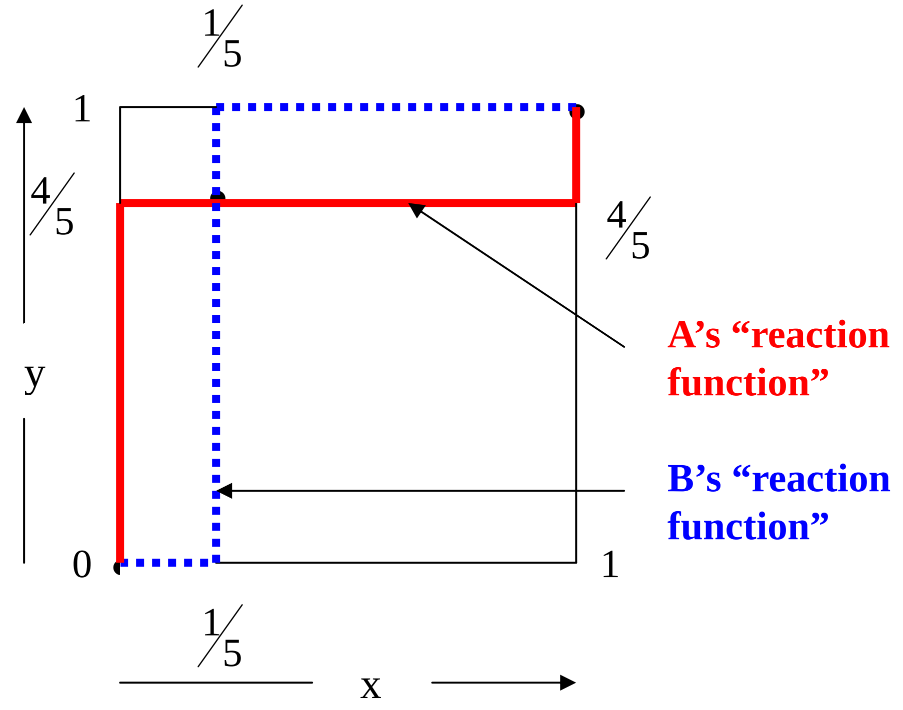
To max. E_A

If $y < 4/5 \rightarrow x = 0$

If $y = 4/5 \rightarrow 0 \leq x \leq 1$

If $y > 4/5 \rightarrow x = 1$

Graphically:



<u>B's perspective</u>

Given: $E_B = y(5x-1) + \underbrace{1 - x}_{\text{independent}}$

To max. E_B

If $x < {}^{1}\!/_{5} \rightarrow y = 0$

If $x = {}^{1}\!/_{5} \rightarrow 0 \leq y \leq 1$

If $x > {}^{1}\!/_{5} \rightarrow y = 1$

So there are three equilibria:

1. $x = 0$, $y = 0$, equilibrium pair: (A_2, B_2), solution value $V(4, 1)$

2. $x = 1$, $y = 1$, equilibrium pair: (A_1, B_2), solution value $V(1, 4)$

3. $x = {}^{1}\!/_{5}$, $y = {}^{4}\!/_{5}$, equilibrium pair:

$(({}^{1}\!/_{5},\ {}^{4}\!/_{5})\,({}^{4}\!/_{5},\ {}^{1}\!/_{5}))$

What is the solution value V(?)

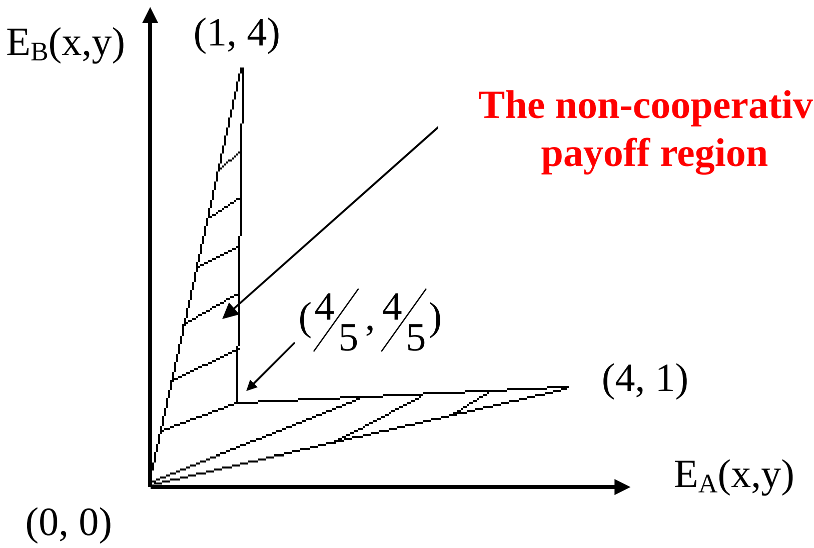
Use the E_A and E_B functions.

$$E_A = \underset{\substack{\uparrow \\ {}^{1}\!/_{5}}}{x}\,(5\underset{\substack{\nwarrow \\ {}^{4}\!/_{5}}}{y} - 4) + 4 - 4\underset{\substack{\uparrow \\ {}^{4}\!/_{5}}}{y}$$

$E_B =$ ……………………..

$V({}^{4}\!/_{5},\ {}^{4}\!/_{5})$

We can plot the multiple equilibria in a graph of $E_A - E_B$ space:



This non-cooperative payoff region is obtained by varying the values of x and y randomly, i.e. with co-ordination between A and B.

3. The prisoner's dilemma: socially suboptimal equilibrium

The "prisoner's dilemma" is the most famous example in variable sum non-cooperative games. In general form, it has the form of the following payoff matrix:

$$\begin{array}{|cc} (c, c) & (a, d) \\ (d, a) & (b, b) \end{array} \quad \text{where } a > b > c > d$$

or, alternatively

$$\begin{tikzcd} (c, c) \arrow[dr, leftrightarrow] & (a, d) \arrow[dl, leftrightarrow] \\ (d, a) & (b, b) \end{tikzcd} \Rightarrow \begin{array}{|cc} (b, b) & (d, a) \\ (a, d) & (c, c) \end{array}$$

still a > b > c > d.

e.g. Two prisoners, A and B, who have jointly committed a crime, are interrogated separately by the police. For both, strategy 1 is to admit the crime, and strategy 2 is not to admit. The payoffs are the numbers of years in prison (so the more the worse). The police have insufficient evidence. Hence if both A and B do not admit the crime, i.e. (A_2, B_2), the court can only sentence each to 2 years of imprisonment.

Then the police tell each of them: if he admits the crime and the other does not, there will already be sufficient evidence. The police will ask for mercy for him in the court and condemn the other. He will get one year and the other will get four years in jail. And vice versa:

	B_1	B_2
A_1	(3, 3)	(1, 4)
A_2	(4, 1)	(2, 2)

We can also solve the prisoner's dilemma by the SWASTIKA method. Remember that the payoffs represent number of years of imprisonment, so the objective of both is to **MINIMIZE** his payoff. Look at matrix 1

$$E_A = 3xy + 1x(1-y) + 4(1-x)y + 2(1-x)(1-y)$$
$$= 2 - x + 2y \qquad \text{--------------------- (1)}$$

$$E_B = 3xy + 4x(1-y) + 1(1-x)y + 2(1-x)(1-y)$$
$$= 2 + 2x - y \qquad \text{--------------------- (2)}$$

From (1) and (2), it is clear that A cannot do anything to offset y in minimizing E_A, nor can B do anything to offset x in minimizing E_B.

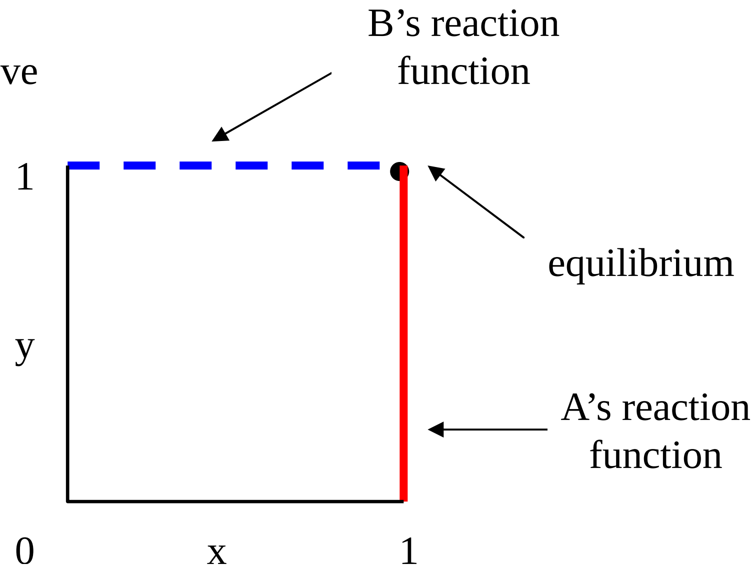
So, given - x in (1); min $E_A \rightarrow x = 1$

Given -y in (2); min $E_B \rightarrow y = 1$

Hence **the equilibrium pair is** (A_1, B_1) and V(3, 3) is the value of the game, which is **socially suboptimal**.

Check that (A_2, B_2) with V(2, 2) are socially optimal, but not equilibrium solutions.

Graphically, we have



The prisoner's dilemma shows that in a situation of social conflict, it is possible to result in a suboptimal equilibrium, from which both parties, motivated by self interest, cannot escape.

One interesting question is this: if A and B are allowed to discuss and "collude", can they escape the "dilemma"?

The answer is no! Why?

The **solution to restore social optimality** may then be:

(1) **external intervention**: morality, law, government policy or force.

(2) **"learning to cooperate"** if the game is repeated infinite number of times". See Varian, chapter 28, pp. 496-498.

Public goods, democracy & the voting paradox

Our discussions about the prisoner's dilemma in general and the under-provision of public goods in specific point to the suboptimality of non-cooperation where there is an interdependence of strategies and welfare. One possible implication is that Pareto improvement in welfare has to be "externally enforced", by a dictator, a moral code, or more moderately in the economic context, by a government.

But how does a government make its decisions about public goods? In many modern societies, the government is democratically elected and **voting** is widely asserted to be a more ideal alternative (of resource allocation) to the **market mechanism** (by democrats and socialists alike). But is such an assertion valid? What is the relationship between democracy and welfare economics? This is a vast subject. Here we can only briefly go over some key areas of research and controversies.

References:

(1) David A. Starrett, Foundations of Public Economics, Cambridge University Press, 1988, Chapter 2.
(2) Hans, van den Doel, Democracy and Welfare Economics, Cambridge University Press, 1979.

(1) The Voting Paradox and the Impossibility Theorem – "A Child's Guide".

Will democracy, or voting (Greek style?) necessarily guarantee Pareto-optimal outcomes? Putting it in another way, will voting turn a game into a co-operative one and ensure a socially optimal solution?

Unfortunately (for democrats), the answer seems to be: NO !

Let us look at the simple example used by Starrett (a student of the great Kenneth Arrow) in his Table 2.1 (p.16)

	X^1	X^2	X^3
A	3	1	2
B	2	3	1
C	1	2	3

There are three voters: A, B and C; and they have three choices: X^1, X^2, X^3.

The figures in the table represent incomes/utility: the more the better. Now if voting is carried out pairwise (i.e. voting on any two choices first, then voting on the selected choice and the remaining third choice), a **paradox** will emerge because

- X^2 beats X^1 (B & C against A; score 2:1)
- X^3 beats X^2 (A & C against B; score 2:1)
- X^1 beats X^3 (A & B against C; score 2:1)

* **So majority voting results in collective intransitivity.** Or, it violates the basic axiom of transitivity of preferences.

Arrow's **"impossibility theorem"**, put in a nutshell, simply states that if the method of summation of individual preferences follows some reasonable conditions (basically five), it is **impossible to assure that the community's decision will not be paradoxical**, just like the example above.

The paradox has extremely serious economic and social implications:

1. The ultimate solution depends on **the voting agenda**! If the agenda in the "parliament" starts with the pair (X^1, X^2), X^2 will be initially chosen, but then beaten by X^3, so X^3

will be ultimate winner. But if the agenda starts with (X^2, X^3), X^3 will have to be compared with X^1. Then X^1 will be chosen. If we start with (X^1, X^3) first, then X^2 will be the final choice! It all depends!

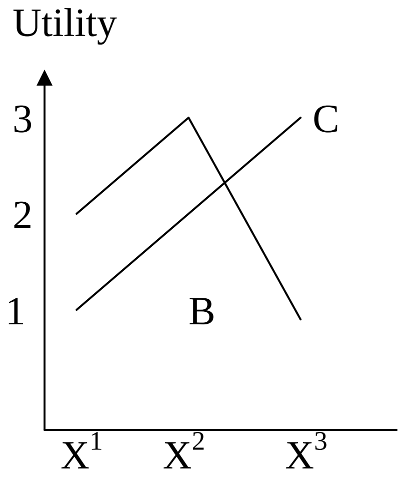
* Any one who controls the agenda controls the outcome! van den Doel even goes so far as to link this result to Robert Michels', "iron law of oligarchy" -- the fatal decline of any democracy into an authoritarian regime where the "agenda" is controlled by the "leadership". (van den Doel, pp.77-78)

2. Such a situation would induce **"cheating"** from voters e.g. if you are voter C, and the agenda starts with (X^2, X^3). If you are selfish and care only for yourself, what should you do (assuming others don't cheat)?
   Answer: Starrett, p.17, 3rd para.

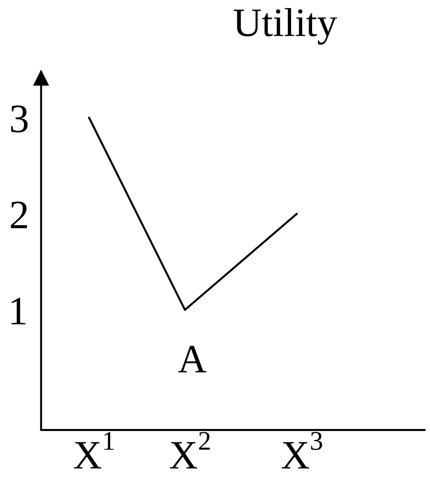
(2) Overcoming the paradox: A.K. Sen's "value–restrictedness" and "single-peaked" order of preferences and political culture. (Another Child's Guide)

The above voting paradox emerges because the preferences of A, B, and C, are not restricted. Moreover, some parties' preferences are "single-peaked" while others' are "non-single-peaked". To overcome the paradox, we can **restrict** all voters' order of preferences to be "single-peaked".

First, let us distinguish between "single-peakedness" and "non-single-peakedness" in preferences : van den Doel (Fig. 4.2, pp.80-81) gives his own example; but here I will convert Starrett's Table 2.1 into the following diagrams to show the differences:



Single-peaked



Non-single-peaked

So the paradox arises because B's and C's orders of preferences are single-peaked, but A's is non-single-peaked.

* However, don't blame A!

His order turns out to be non-single-peaked because we line up the choices in the sequence of X^1, X^2, X^3. (What if we sequence as follows: X^2, X^3 and X^1? Who's to "blame" then?)

Answer: B.

His order would then be non-single-peaked.

In any case, suppose we sequence in the manner (X^1, X^2, X^3) (which reflects our "ideological bias" regarding X^1 and X^3 as "extremes"), then the "trouble-maker" is A, who unwittingly prefers to vote for "extremes", i.e. X^1 or X^3 while B and C, being, "moderate", "responsible" citizens show consistency.

To dramatize, imagine the following groups of people.

B: Middle class people, who always prefer middle-of-the-road policies.

C: Capitalists, who prefer X^3 (free market economics) to X^2

(mixed economics) and to X^1 (heavy government intervention or even socialist planning).

If the voters are composed of the B type and the C type, then there will be no serious problems (and certainly no paradoxes): the solution is somewhere between X^2 and X^3. But comes A:

A: Revolutionary radicals, who prefer X^1 (planning) or X^3 (free market) because X^3 will increase class conflict and lead to a revolution! The mass will rise up and overthrow the capitalists. So, A can get X^3 indirectly!

Because of A, inconsistency and paradox would arise. (Should we then take away A's voting right?)

That solution is probably too undemocratic. What if a person like A becomes "moderate", giving up his/her "radicalism" and his order of preferences becomes "single-peaked"? To make him "consistent", "non-extremist", I proceed to modify Starrett's Table 2.1 of into the following:

My Table

	X^1	X^2		X^3
A	3	2	⟶	1
B	2	3	⟵	1
C	1	2		3

The arrows represent the changes c.f. Starrett's original Table 2.1

It is obvious that A's preferences are also single-peaked now (although his inclination is the opposite to C's: the capitalist's, but A is now "non-extremist", he is "true" to his conviction: heavy government intervention/planning.

*** Now with the above My Table: **the voting paradox disappears.** Arrow's theorem is laid to rest. Check:

X^2 beats X^1 (B & C against A; 2:1)
X^2 beats X^3 (A & B against C; 2:1)

So $\underline{X^2}$ will emerge as the ultimate choice of the society under the system of majority voting.

Another check:

The agenda starts with (X^1, X^3) X^1 beats X^3 (A & B against C) but X^1 is again beaten by X^2 → solution $\underline{X^2}$ becomes invariant.

Hence, how are the voting paradox, the impossibility theorem and the like overcome? The answer is: "value-restrictedness" or "single-peakedness in the order of preferences" or "non-extremism".

All these lead to the theory of the median voter. The implications are also serious: it implies that a democracy will be optimal if the political culture is mature and based on "non-extremism". I would quote van den Doel on this point (pp.81-82):

"......... political culture serves to make working of the majority rule possible. Single-peakedness of all preference orderings means that there is a cultural agreement about the criteria on the basis of which a decision must be taken even though there may be disagreement about the decision itself. For example, all voters vote for the party which approximates their own opinions most closely; nobody votes for a party which is furthest removed from his views. Value-restrictedness means among other things that some alternatives, even though they are being discussed, are ranked by no one as the best. For instance: force is only accepted as the last resort. Both single-peakedness and value-restrictedness mirror a cultural consensus without which democracy cannot function."

- **So, democracy, majority voting etc. will be welfare-enhancing in a society that is relatively stable and not torn by class conflicts?**
- What if preferences are not so restricted?
  e.g. China in 1920's-30's?